Another approach to the fundamental theorem of Riemannian geometry in R3, by way of rotation fields

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ANOTHER APPROACH TO THE FUNDAMENTAL THEOREM OF RIEMANNIAN GEOMETRY IN $\mathbb{R}^3$, BY WAY OF ROTATION FIELDS

P.G. CIARLET, L. GRATIE, O. IOSIFESCU, C. MARDARE, C. VALLÉE

Abstract. In 1992, C. Vallée showed that the metric tensor field $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ associated with a smooth enough immersion $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ defined over an open set $\Omega \subset \mathbb{R}^3$ necessarily satisfies the compatibility relation

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \Omega,$$

where the matrix field $\boldsymbol{\Lambda}$ is defined in terms of the field $\mathbf{U} = \mathbf{C}^{1/2}$ by

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\Big\{\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T])\mathbf{U}\Big\}.$$

The main objective of this paper is to establish the following converse: If a smooth enough field $\mathbf{C}$ of symmetric and positive-definite matrices of order three satisfies the above compatibility relation over a simply-connected open set $\Omega \subset \mathbb{R}^3$, then there exists, typically in spaces such as $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ or $\mathcal{C}^2(\Omega;\mathbb{R}^3)$, an immersion $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ such that $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ in Ω.

This global existence theorem thus provides an alternative to the fundamental theorem of Riemannian geometry for an open set in $\mathbb{R}^3$, where the compatibility relation classically expresses that the Riemann curvature tensor associated with the field $\mathbf{C}$ vanishes in Ω.

The proof consists in first determining an orthogonal matrix field $\mathbf{R}$ defined over Ω, then in determining an immersion $\boldsymbol{\Theta}$ such that $\boldsymbol{\nabla\Theta} = \mathbf{R}\mathbf{C}^{1/2}$ in Ω, by successively solving two Pfaff systems. In addition to its novelty, this approach thus also possesses a more "geometrical" flavor than the classical one, as it directly seeks the polar factorization $\boldsymbol{\nabla\Theta} = \mathbf{RU}$ of the immersion gradient in terms of a rotation $\mathbf{R}$ and a pure stretch $\mathbf{U} = \mathbf{C}^{1/2}$. This approach also constitutes a first step towards the analysis of models in nonlinear three-dimensional elasticity where the rotation field is considered as one of the primary unknowns.

Résumé. En 1992, C. Vallée a montré que le champ $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ de tenseurs métriques associé à une immersion suffisamment régulière $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ définie sur un ouvert $\Omega \subset \mathbb{R}^3$ vérifie nécessairement la relation de compatibilité

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \Omega,$$

où le champ $\boldsymbol{\Lambda}$ de matrices est défini en fonction du champ $\mathbf{U} = \mathbf{C}^{1/2}$ par

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\Big\{\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T])\mathbf{U}\Big\}.$$

L'objet principal de cet article est d'établir la réciproque suivante: Si un champ suffisamment régulier $\mathbf{C}$ de matrices symétriques définies positives d'ordre trois satisfait la relation de compatibilité ci-dessus dans un ouvert $\Omega \subset \mathbb{R}^3$ simplement connexe, alors il existe, typiquement dans des espaces tels que $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ ou $\mathcal{C}^2(\Omega;\mathbb{R}^3)$, une immersion $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ telle que $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ in Ω.

Ce théorème d'existence global fournit donc une alternative au théorème fondamental de la géométrie riemannienne pour un ouvert $\Omega \subset \mathbb{R}^3$, dans lequel la relation de compatibilité exprime classiquement que le tenseur de courbure de Riemann associé au champ $\mathbf{C}$ s'annule dans Ω.

La démonstration consiste d'abord à déterminer un champ $\mathbf{R}$ de matrices orthogonales dans Ω, puis à déterminer une immersion $\boldsymbol{\Theta}$ telle que $\boldsymbol{\nabla\Theta} = \mathbf{R}\mathbf{C}^{1/2}$ dans Ω, en résolvant successivement deux systèmes de Pfaff. En plus de sa nouveauté, cette approche est donc de nature plus "géométrique" que l'approche classique, dans la mesure où elle cherche à identifier directement la factorisation polaire $\boldsymbol{\nabla\Theta} = \mathbf{RU}$ du gradient de l'immersion en une rotation $\mathbf{R}$ et une extension pure $\mathbf{U} = \mathbf{C}^{1/2}$.

Cette approche constitue également un premier pas vers l'analyse de modèles en élasticité tridimensionnelle non linéaire où le champ des rotations est considéré comme une inconnue à part entière.

Key words and phrases. Classical differential geometry, fundamental theorem of Riemannian geometry, Pfaff systems, polar factorization, nonlinear three-dimensional elasticity.

1. Introduction

All the notions and notations used, but not defined, in this introduction are defined in the next section.

Latin indices range in the set $\{1, 2, 3\}$. Let $\mathbb{S}^3_>$ designate the set of all real symmetric and positive-definite matrices of order three. Let Ω be an open subset of $\mathbb{R}^3$ and let $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ be an immersion. The *metric tensor field* $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ of the manifold $\mathbf{\Theta}(\Omega)$, considered as isometrically imbedded in $\mathbb{R}^3$, is then defined by $\mathbf{C} := \boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta}$.

It is well known that the matrix field $\mathbf{C} = (g_{ij})$ defined in this fashion cannot be arbitrary. More specifically, let

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq},$$

where $(g^{pq}) := (g_{ij})^{-1}$. Then *the functions g_{ij} necessarily verify the compatibity relations*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega,$$

which in effect simply constitute a re-writing of the relations $\partial_{ikj}\mathbf{\Theta} = \partial_{kij}\mathbf{\Theta}$. The functions Γ_{ijq} and Γ^p_{ij} are the *Christoffel symbols of the first and second kinds*, and the functions R_{qijk} are the covariant components of the *Riemann curvature tensor field*, associated with the immersion $\mathbf{\Theta}$.

It is also well known that, conversely, if a matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfies the relations $R_{qijk} = 0$ in a *simply-connected* open subset Ω of $\mathbb{R}^3$, the functions Γ_{ijq}, Γ^p_{ij}, and R_{qijk} being then defined as above from the functions g_{ij}, then *there exists an immersion* $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ *such that*

$$\mathbf{C} = \boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta} \text{ in } \Omega.$$

If the set Ω is in addition *connected,* such an immersion is *uniquely defined up to isometries of* $\mathbb{R}^3$. This means that any immersion $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ satisfying $\mathbf{C} = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^T \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}$ in Ω is necessarily of the form $\widetilde{\mathbf{\Theta}} = \mathbf{a} + \mathbf{Q}\mathbf{\Theta}$, with $\mathbf{a}$ a vector in $\mathbb{R}^3$ and $\mathbf{Q}$ an orthogonal matrix of order three. Among all such immersions $\widetilde{\mathbf{\Theta}}$, some are therefore *orientation-preserving*, i.e., they satisfy $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} > 0$ in Ω.

Otherwise, the immersion $\mathbf{\Theta}$ becomes *uniquely defined* if the following additional *"initial" conditions*:

$$\mathbf{\Theta}(x_0) = \mathbf{a}_0 \text{ and } \boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{F}_0,$$

are imposed, where x_0 is any point in Ω, $\mathbf{a}_0$ is any vector in $\mathbb{R}^3$, and $\mathbf{F}_0$ is any matrix of order three that satisfies $\mathbf{F}_0^T\mathbf{F}_0 = \mathbf{C}(x_0)$, for instance $\mathbf{F}_0 = \mathbf{C}(x_0)^{1/2}$ (for self-contained, and essentially elementary, proofs of these classical existence and uniqueness results, see Ciarlet & Larsonneur [5] or Ciarlet [3, Chapter 1]).

The above *regularity assumption* on the symmetric and positive-definite matrix field $\mathbf{C}$ can be substantially weakened. In this direction, C. Mardare [13] has shown that the existence theorem still holds if $\mathbf{C} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$, with a resulting immersion $\mathbf{\Theta}$ in the space $\mathcal{C}^2(\Omega, \mathbb{R}^3)$. Then S. Mardare [15] further improved this result, by showing that the existence theorem again still holds if $\mathbf{C} \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{S}^3_>)$, with a resulting mapping $\mathbf{\Theta}$ in the space $W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^3)$. Naturally, the sufficient (and clearly necessary) relations $R_{qijk} = 0$ are then assumed to hold only in the sense of distributions, viz., as

$$\int_\Omega \{-\Gamma_{ikq}\partial_j\varphi + \Gamma_{ijq}\partial_k\varphi + \Gamma^p_{ij}\Gamma_{kqp}\varphi - \Gamma^p_{ik}\Gamma_{jqp}\varphi\}dx = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$. If the simply-connected open set Ω is in addition connected, the mappings $\boldsymbol{\Theta}$ found in [13] and [15] are again uniquely defined up to isometries of $\mathbb{R}^3$, or they again become uniquely defined if the above initial conditions are imposed at some point $x_0 \in \Omega$.

Note that all the above existence and uniqueness theorems hold *verbatim* in $\mathbb{R}^d$ for any dimension $d \geq 2$.

Let $\mathbb{M}^3$ and $\mathbb{O}^3$ respectively designate the space of all real matrices of order three and the set of all real orthogonal matrices of order three. Let again $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ be an immersion defined on an open subset Ω of $\mathbb{R}^3$. In 1992, Vallée [20] has shown that a *different* set of *necessary compatibility relations* is also satisfied by the metric tensor field $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ associated with the immersion $\boldsymbol{\Theta}$.

C. Vallée's key idea was to make use of the *polar factorization*

$$\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{R}\mathbf{U}$$

of the matrix field $\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$; this means that $\mathbf{R} \in \mathcal{C}^2(\Omega; \mathbb{O}^3)$ is the orthogonal matrix field, and $\mathbf{U} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ is the symmetric and positive-definite symmetric matrix field, respectively defined by

$$\mathbf{R} := \boldsymbol{\nabla}\boldsymbol{\Theta}\mathbf{C}^{-1/2} \text{ and } \mathbf{U} := \mathbf{C}^{1/2}.$$

Note that, if $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in Ω and the mapping $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ is injective, in which case $\boldsymbol{\Theta}$ may be thought of as a *deformation* of a continuum, this polar factorization is nothing but the classical decomposition at each point $x \in \Omega$ of the *deformation gradient* $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ into a *rotation* represented by the proper orthogonal matrix $\mathbf{R}(x)$, and into a *pure stretch* represented by the matrix $\mathbf{U}(x)$. In this sense, C. Vallée's approach is more "geometrical" than the classical one, as it makes an essential use of the "local geometry of a deformation" by means of the fields $\mathbf{R}$ and $\mathbf{U}$.

First, C. Vallée shows that the orthogonality of the matrix field $\mathbf{R} \in \mathcal{C}^2(\Omega; \mathbb{O}^3)$ implies that there exists a matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ such that, at each point $x \in \Omega$,

$$(D\mathbf{R}(x)\mathbf{a})\mathbf{b} = \mathbf{R}(x)(\boldsymbol{\Lambda}(x)\mathbf{a} \wedge \mathbf{b}) \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3,$$

where $D\mathbf{R}(x) \in L(\mathbb{R}^3; \mathbb{M}^3)$ denotes the Fréchet derivative at $x \in \Omega$ of the mapping $\mathbf{R} : \Omega \to \mathbb{M}^3$. C. Vallée also shows that the relations $\partial_{ij}\mathbf{R} = \partial_{ji}\mathbf{R}$ in Ω imply furthermore that *the matrix field* $\boldsymbol{\Lambda}$ *necessarily satisfies the compatibility relation*

$$\mathbf{CURL}\ \boldsymbol{\Lambda} + \mathbf{COF}\ \boldsymbol{\Lambda} = \mathbf{0} \text{ in } \Omega.$$

It is to be emphasized that the existence of such a matrix field $\boldsymbol{\Lambda}$ and the above compatibility relation satisfied by $\boldsymbol{\Lambda}$ both hold for *any* orthogonal field $\mathbf{R} \in \mathcal{C}^2(\Omega; \mathbb{O}^3)$, i.e., regardless of the particular form, viz. $\mathbf{R} = \boldsymbol{\nabla}\boldsymbol{\Theta}\mathbf{C}^{-1/2}$, that it assumes here.

Second, taking now into account that the field $\mathbf{R}$ is of the specific form $\mathbf{R} = \boldsymbol{\nabla}\boldsymbol{\Theta}\mathbf{U}^{-1}$ with $\mathbf{U} = \mathbf{C}^{1/2}$, and using the relations $\partial_{kl}\boldsymbol{\Theta} = \partial_{lk}\boldsymbol{\Theta}$, C. Vallée shows that *the matrix field* $\boldsymbol{\Lambda}$ *is given by*

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}} \Big\{ \mathbf{U}\ (\mathbf{CURL}\ \mathbf{U})^T \mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}\ (\mathbf{CURL}\ \mathbf{U})^T])\mathbf{U} \Big\}.$$

The relation $\mathbf{CURL}\ \boldsymbol{\Lambda} + \mathbf{COF}\ \boldsymbol{\Lambda} = \mathbf{0}$*, with* $\boldsymbol{\Lambda}$ *replaced by this expression in terms of* $\mathbf{U} = \mathbf{C}^{1/2}$*, thus constitutes another compatibility relation that a matrix field* $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *necessarily satisfies if it is of the form* $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ *for some immersion* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$.

Note that this compatibility relation is solely expressed in terms of the matrix field $\mathbf{C}$, by way of its square root $\mathbf{U} = \mathbf{C}^{1/2}$, hence *without any recourse to the Christoffel symbols* as in the classical relations $R_{qijk} = 0$. Note also that it is the same *Schwarz lemma* that is the keystone for both kinds of compatibility relations, either in the form of the relations

$\partial_{ij}\mathbf{R} = \partial_{ji}\mathbf{R}$ and $\partial_{kl}\mathbf{\Theta} = \partial_{lk}\mathbf{\Theta}$ as here, or in form of the relations $\partial_{ikj}\mathbf{\Theta} = \partial_{kij}\mathbf{\Theta}$ used for deriving the relations $R_{qijk} = 0$. In the same spirit, the cancellation of both the curvature and the torsion, expressed in the classical approach by means of the relations $R_{qijk} = 0$ and $\Gamma^p_{ij} = \Gamma^p_{ji}$ in Ω, likewise manifest themselves in C. Vallée's approach, *albeit* in a more subtle way; in this respect, see Hamdouni [12].

The main objective of this paper is to show that, *conversely, if a matrix field* $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *satisfies*

$$\mathbf{CURL}\,\mathbf{\Lambda} + \mathbf{COF}\,\mathbf{\Lambda} = \mathbf{0}$$

in a simply-connected open subset Ω *of* $\mathbb{R}^3$, *with* $\mathbf{\Lambda}$ *having the above expression in terms of* $\mathbf{U} = \mathbf{C}^{1/2}$, *then there exists an immersion* $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ *such that*

$$\mathbf{C} = \mathbf{\nabla\Theta}^T \mathbf{\nabla\Theta} \text{ in } \Omega$$

(cf. Theorems 6.1 and 6.2).

This result is itself a consequence of the following general existence theorem (cf. Theorem 5.1; in fact this general result holds *verbatim* in $\mathbb{R}^d$ for an arbitrary dimension $d \geq 2$ but, for coherence, it is enunciated here only for $d = 3$): Let $\mathbb{A}^3$ designate the space of all real antisymmetric matrices of order three. *Let* Ω *be a connected and simply-connected open subset of* $\mathbb{R}^3$ *and let there be given a symmetric and positive-definite matrix field* $\mathbf{U} \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{S}^3_>)$ *that satisfies the relations*

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_j \mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{A}^3),$$

where the matrix fields $\mathbf{A}_i \in L^\infty_{\text{loc}}(\Omega; \mathbb{A}^3)$ *are defined in terms of* $\mathbf{U}$ *by*

$$\mathbf{A}_j := \frac{1}{2}(\mathbf{U}^{-1}(\mathbf{\nabla c}_j - (\mathbf{\nabla c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j \mathbf{U} - (\partial_j \mathbf{U})\mathbf{U}^{-1},$$

the notation $\mathbf{c}_j$ designating the j-th column vector field of the matrix field $\mathbf{U}^2$. *Then there exists an immersion* $\mathbf{\Theta} \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^3)$ *such that*

$$\mathbf{U}^2 = \mathbf{\nabla\Theta}^T \mathbf{\nabla\Theta} \text{ in } \Omega,$$

and $\mathbf{\Theta}$ *is uniquely defined up to isometries of* $\mathbb{R}^3$ (the above relations are also necessarily satisfied by any given immersion $\mathbf{\Theta} \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^3)$, even if Ω is not simply-connected; cf. Theorem 3.1).

The proof consists *first* in determining an orthogonal matrix field $\mathbf{R} \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{O}^3)$ by solving the *Pfaff system* $\partial_i \mathbf{R} = \mathbf{R}\mathbf{A}_i$ in Ω, *then* in determining the immersion $\mathbf{\Theta} \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^3)$ by solving the equation $\mathbf{\nabla\Theta} = \mathbf{RU}$ in Ω. By contrast, the proof in the "classical" approach consists in first determining a matrix field $\mathbf{F} = (F_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^3)$ by solving the Pfaff system $\partial_i F_{lj} = \Gamma^p_{ij} F_{lp}$ in Ω, then in determining the immersion $\mathbf{\Theta} \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^3)$ by solving the equation $\mathbf{\nabla\Theta} = \mathbf{F}$ in Ω.

The above compatibility relations satisfied by the matrix fields $\mathbf{A}_j$ were first noticed by Shield [17], who was also the first to recognize the importance of the polar factorization $\mathbf{\nabla\Theta} = \mathbf{RU}$ for deriving necessary compatibility relations that the matrix field $\mathbf{C} = \mathbf{\nabla\Theta}^T \mathbf{\nabla\Theta}$ satisfies. In this direction, see also Pietraszkiewicz & Badur [16], who further elaborated on this idea in the context of continuum mechanics.

In addition, R. T. Shield pointed out that these relations are also sufficient for the existence of an immersion in spaces of continuously differentiable functions. Using the techniques of exterior differential calculus, Edelen [8] likewise noticed that the recovery of the immersion $\mathbf{\Theta}$ could be also achieved through the recovery of an orthogonal matrix field.

The core of our argument consists in showing (see the proof of Theorem 6.1) that *the compatibility relations satisfied by the above matrix fields* $\mathbf{A}_i$ *are in fact equivalent to those* *proposed by C. Vallée* (the key link consists in defining the matrix field $\mathbf{\Lambda}$ by letting its j-th column vector field $\mathbf{a}_j$ be such that $\mathbf{A}_j\mathbf{b} = \mathbf{a}_j \wedge \mathbf{b}$ for all $\mathbf{v} \in \mathbb{R}^3$), thus demonstrating the sufficiency of C. Vallée's compatibility relations.

We emphasize that our existence results (Theorems 5.1 and 6.1) are *global* and that they hold in function *spaces "with little regularity"*, viz., $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$, thanks essentially to a deep global existence theorem for Pfaff systems (recalled in Theorem 4.1), also with little regularity, recently established by S. Mardare [14]. Note that we also obtain global existence theorems in the spaces $C^2(\Omega;\mathbb{R}^3)$ and $W^{2,\infty}(\Omega;\mathbb{R}^3)$ (Theorems 5.2 and 5.3 and Theorems 6.2 and 6.3).

As advocated notably by Fraeijs de Veubeke [9], Pietraszkiewicz & Badur [16], or Simo & Marsden [18], *rotation fields* can be introduced as *bona fide* unknowns in nonlinear three-dimensional elasticity. This introduction typically involves the replacement of the deformation gradient $\boldsymbol{\nabla\Theta}$ in the stored energy function by a *rotation field* and a *pure stretch field* $\mathbf{U}$, "the constraint" $\boldsymbol{\nabla\Theta} = \mathbf{RU}$ being enforced by means of an appropriate *Lagrange multiplier*, thus producing a *multi-field variational principle*.

The existence theory for models based on such principles appears to be an essentially virgin territory (with the noticeable exception of Grandmont, Maday & Métier [10], who considered a time-dependent elasticity problem in dimension two where a "global rotation" is one of the unknowns). It is thus hoped that the present work constitutes a first, yet admittedly small, step towards the mathematical analysis of such models.

The results of this paper have been announced in [4].

2. Notations and preliminaries

This section gathers various conventions, notations, and definitions, as well as some preliminary results, that will be used in this article.

In Sections 2 to 5, the notation p designates any integer ≥ 2. It is then understood that Latin indices and exponents range in the set $\{1, 2, \cdots, p\}$ and that the summation convention with respect to repeated indices and exponents is used in conjunction with this rule. In Section 6, the same rules apply with $p = 3$.

All matrices considered in this paper have real elements. The notations $\mathbb{M}^p$, $\mathbb{S}^p$, $\mathbb{S}^p_>$, $\mathbb{A}^p$, and $\mathbb{O}^p$ respectively designate the sets of all square matrices, of all symmetric matrices, of all symmetric and positive-definite symmetric matrices, of all antisymmetric matrices, and of all orthogonal matrices, of order p. The notation $\mathbb{M}^{p\times q}$ designates the set of all matrices with p rows and q columns. The notation (a_{ij}) designates the matrix in $\mathbb{M}^p$ with a_{ij} as its elements, the first index being the row index, and given a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{M}^p$, the notation $(\mathbf{A})_{ij}$ designates its element a_{ij}. When it is identified with a matrix, a vector in $\mathbb{R}^p$ is always understood as a column vector, i.e., a matrix in $\mathbb{M}^{p\times 1}$.

The Euclidean norm of $\mathbf{a} \in \mathbb{R}^p$ is denoted $|\mathbf{a}|$ and the Euclidean inner-product of $\mathbf{a} \in \mathbb{R}^p$ and $\mathbf{b} \in \mathbb{R}^p$ is denoted $\mathbf{a} \cdot \mathbf{b}$.

The vector product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ is denoted $\mathbf{a} \wedge \mathbf{b}$. The *cofactor matrix* $\mathbf{COF}$ $\mathbf{A}$ associated with a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{M}^3$ is defined by

$$\mathbf{COF\ A} := \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{32}a_{13} - a_{33}a_{12} & a_{33}a_{11} - a_{31}a_{13} & a_{31}a_{12} - a_{32}a_{11} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

Given any matrix $\mathbf{C} \in \mathbb{S}^p_>$, there exists a unique matrix $\mathbf{U} \in \mathbb{S}^p_>$ such that $\mathbf{U}^2 = \mathbf{C}$ (for a proof, see, e.g., Ciarlet [2, Theorem 3.2-1). The matrix $\mathbf{U}$ is denoted $\mathbf{C}^{1/2}$ and is called the *square root* of $\mathbf{C}$. The mapping $\mathbf{C} \in \mathbb{S}^p_> \to \mathbf{C}^{1/2} \in \mathbb{S}^p_>$ defined in this fashion is of class $\mathbf{C}^\infty$ (for a proof, see, e.g., Gurtin [11, Section 3]).

Any invertible matrix $\mathbf{F} \in \mathbb{M}^p$ admits a unique *polar factorization* $\mathbf{F} = \mathbf{RU}$, as a product of a matrix $\mathbf{R} \in \mathbb{O}^p$ by a matrix $\mathbf{U} \in \mathbb{S}^p_>$, with $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}$ and $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ (the uniqueness of such a factorization follows from the uniqueness of the square root of a matrix $\mathbf{C} \in \mathbb{S}^p_>$).

The coordinates of a point $x \in \mathbb{R}^p$ are denoted x_i. Partial derivative operators, in the usual sense or in the sense of distributions, of the first, second, and third order are denoted $\partial_i := \partial/\partial x_i, \partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

All the vector spaces considered in this paper are over $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^p$. The notation $U \Subset \Omega$ means that $\overline{U}$ is a compact subset of Ω. The notations $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ respectively designate the space of all functions in $\mathcal{C}^\infty(\Omega)$ whose support is compact and contained in Ω, and the space of distributions over Ω. The notations $\mathcal{C}^l(\Omega), l \geq 0$, and $W^{m,\infty}(\Omega), m \geq 0$, respectively designate the spaces of continuous functions over Ω for $l = 0$, or l-times continuously differentiable functions over Ω for $l \geq 1$, and the usual Sobolev spaces, with $W^{o,\infty}(\Omega) = L^\infty(\Omega)$. Finally, $W^{m,\infty}_{\mathrm{loc}}(\Omega)$ designates the space of equivalent classes of measurable functions on Ω whose restriction to any open set $U \Subset \Omega$ belongs to the space $W^{m,\infty}(U)$.

Let $\mathbb{X}$ be any finite-dimensional space, such as $\mathbb{R}^p, \mathbb{M}^{p\times q}, \mathbb{A}^p$, etc., or a subset thereof, such as $\mathbb{S}^p_>, \mathbb{O}^d$, etc. Then notations such as $\mathcal{D}'(\Omega;\mathbb{X})$, $\mathcal{C}^l(\Omega;\mathbb{X})$, $L^\infty_{loc}\Omega;\mathbb{X})$, etc., designate spaces or sets of vector fields or matrix fields with values in $\mathbb{X}$ and whose components belong to $\mathcal{D}'(\Omega)$, $\mathcal{C}^l(\Omega)$, $L^\infty_{\mathrm{loc}}(\Omega)$, etc.

Given a mapping $\mathbf{\Theta} = (\Theta_i) \in \mathcal{D}'(\Omega;\mathbb{R}^p)$, the matrix field $\mathbf{\nabla\Theta} \in \mathcal{D}'(\Omega;\mathbb{M}^p)$ is defined by $(\mathbf{\nabla\Theta})_{ij} = \partial_j\Theta_i$. Given a matrix field $\mathbf{A} = (a_{ij}) \in \mathcal{D}'(\Omega;\mathbb{M}^3)$, the notation $\mathbf{CURL\ A}$ designate the matrix field

$$\mathbf{CURL\ A} := \begin{pmatrix} \partial_2 a_{13} - \partial_3 a_{12} & \partial_3 a_{11} - \partial_1 a_{13} & \partial_1 a_{12} - \partial_2 a_{11} \\ \partial_2 a_{23} - \partial_3 a_{22} & \partial_3 a_{21} - \partial_1 a_{23} & \partial_1 a_{22} - \partial_2 a_{21} \\ \partial_2 a_{33} - \partial_3 a_{32} & \partial_3 a_{31} - \partial_1 a_{33} & \partial_1 a_{32} - \partial_2 a_{31} \end{pmatrix} \in \mathcal{D}'(\Omega;\mathbb{M}^3).$$

Although the elements in the spaces $L^\infty_{\mathrm{loc}}(\Omega)$ or $W^{m,\infty}_{\mathrm{loc}}(\Omega), m \geq 1$, are equivalence classes of functions, they will be conveniently identified in this article with functions defined over Ω, as follows.

Any equivalence class $F \in L^\infty_{\mathrm{loc}}(\Omega)$ will be identified with the unique element in F that is unambiguously defined at all $x \in \Omega$ by

$$f(x) := \liminf_{\varepsilon\to 0} \left\{ \left(\int_{B(x;\varepsilon)} dy \right)^{-1} \int_{B(x;\varepsilon)} g(y) dy \right\},$$

where $B(x;\varepsilon) := \{y \in \Omega;\ |y - x| < \varepsilon\}$ and g is any element in the equivalence class F. As a result, the point values $f(x)$ of any "function" $f \in L^\infty_{\mathrm{loc}}(\Omega)$ become unambiguously defined for all $x \in \Omega$ (the above inferior limit is always a well-defined real number).

Likewise, any equivalence class in the space $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ will be indentified with a continuous function over Ω, thanks this time to the imbedding $W^{1,\infty}(B) \subset \mathcal{C}^0(\overline{B})$, which holds for any open ball $B \subset \Omega$. The notation $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ thus means that f is the unique continuous function in what is normally an equivalence class. As a result of this identification, the point values $f(x)$ of any "function" $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ likewise become unambiguously defined for all $x \in \Omega$.

Finally, we recall that a mapping $\mathbf{\Theta} \in \mathcal{C}^1(\Omega;\mathbb{R}^p)$ is an *immersion* if the matrix $\mathbf{\nabla\Theta}(x) \in \mathbb{M}^p$ is invertible at all points $x \in \Omega$.

3. Compatibility relations satisfied by the matrix field $\mathbf{U} = (\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta})^{1/2}$

Let Ω be an arbitrary open subset of $\mathbb{R}^p$. As recalled in Section 1, it is well known that the *metric tensor* field $\mathbf{C} := \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega;\mathbb{S}^p_>)$ associated with an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega;\mathbb{R}^p)$ *necessarily* satisfies *compatibility relations* that take the form $R_{qijk} = 0$ in Ω, where the functions $R_{qijk} \in \mathcal{C}^0(\Omega)$ are the covariant components of the associated *Riemann curvature tensor*.

The next theorem shows that, likewise, the matrix field $\mathbf{U} := \mathbf{C}^{1/2}$ *necessarily* satisfies *ad hoc compatibility relations*. These relations, which were first established in componentwise form by Shield [17] for smooth immersions, are established here in more concise matrix form. In addition, they are shown to hold in function spaces with little regularity.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^p$ and let there be given an immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^p)$. At each point $x \in \Omega$, let $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{R}(x)\mathbf{U}(x)$, with*

$$\mathbf{U}(x) := (\boldsymbol{\nabla}\boldsymbol{\Theta}^T(x)\boldsymbol{\nabla}\boldsymbol{\Theta}(x))^{1/2} \in \mathbb{S}^p_> \text{ and } \mathbf{R}(x) := \boldsymbol{\nabla}\boldsymbol{\Theta}(x)\mathbf{U}(x)^{-1} \in \mathbb{O}^p,$$

denote the unique polar factorization of the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$. Then the fields $\mathbf{U}$ and $\mathbf{R}$ defined in this fashion possess the following regularities:

$$\mathbf{U} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>) \text{ and } \mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{O}^p).$$

Let the matrix fields $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega;\mathbb{A}^p)$ be defined in terms of the matrix field $\mathbf{U}$ by

$$\mathbf{A}_j := \frac{1}{2}\{\mathbf{U}^{-1}(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\},$$

where $\mathbf{c}_j \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{R}^p)$ denotes the j-th column vector field of the matrix field $\mathbf{C} := \mathbf{U}^2 \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>)$. Then the matrix field $\mathbf{U}$ necessarily satisfies the following compatibility relations:

$$\partial_i\mathbf{A}_j - \partial_j\mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega;\mathbb{A}^p).$$

Proof. Since both mappings $\mathbf{C} \in \mathbb{S}^p_> \to \mathbf{C}^{1/2} \in \mathbb{S}^p_>$ and $\mathbf{U} \in \mathbb{S}^p_> \to \mathbf{U}^{-1} \in \mathbb{S}^p_>$ are of class $\mathcal{C}^\infty$ and the immersion $\boldsymbol{\Theta}$ belongs to the space $W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^p)$ by assumption, the fields $\mathbf{U}$ and $\mathbf{R}$ clearly possess the announced regularities.

Given an immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^p)$, define the matrix fields

$$\begin{aligned}\mathbf{F} &:= \boldsymbol{\nabla}\boldsymbol{\Theta} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{M}^p),\\ \mathbf{C} = \mathbf{U}^2 = (g_{ij}) &:= \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>),\\ (g^{kl}) &:= (g_{ij})^{-1} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>);\end{aligned}$$

define the vector fields

$$\begin{aligned}\mathbf{g}_l &:= \partial_l\boldsymbol{\Theta} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{R}^p),\\ \mathbf{g}^k &:= g^{kl}\mathbf{g}_l \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{R}^p);\end{aligned}$$

and finally, define the matrix fields

$$\boldsymbol{\Gamma}_j = (\Gamma^k_{jl}) \in L^\infty_{\rm loc}(\Omega;\mathbb{M}^p),$$

where k and l respectively designate the row and column indices and

$$\Gamma^k_{jl} := \partial_j\mathbf{g}_l \cdot \mathbf{g}^k$$

(the functions Γ^k_{jl} are the Christoffel symbols of the first kind).

Then it is easily verified that

$$\partial_j \mathbf{F} = \mathbf{F}\boldsymbol{\Gamma}_j \text{ in } L^\infty_{\rm loc}(\Omega;\mathbb{M}^p).$$

Since the polar factorization $\mathbf{F} = \mathbf{RU}$ implies that $\partial_j \mathbf{F} = \mathbf{RU}\boldsymbol{\Gamma}_j = (\partial_j \mathbf{R})\mathbf{U} + \mathbf{R}\partial_j \mathbf{U}$, and since the matrices $\mathbf{U}(x)$ are invertible for all $x \in \Omega$, we also have

$$\partial_j \mathbf{R} = \mathbf{R}\mathbf{A}_j \text{ in } L^\infty_{\rm loc}(\Omega;\mathbb{M}^p),$$

where

$$\mathbf{A}_j := (\mathbf{U}\boldsymbol{\Gamma}_j - \partial_j \mathbf{U})\mathbf{U}^{-1} \in L^\infty_{\rm loc}(\Omega;\mathbb{M}^p).$$

In what follows, ${}_{X'}< \cdot, \cdot >_X$ designates the duality pairing between a topological space X and its dual X'. For notational conciseveness, spaces such as $\mathcal{D}(\Omega;\mathbb{M}^p), H^1_0(U,\mathbb{M}^p)$, etc., will be abbreviated as $\mathcal{D}(\Omega), H^1_0(U)$, etc., in the duality pairings.

Given a matrix field $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{M}^p)$ and a matrix field $\mathbf{A} \in L^\infty_{\rm loc}(\Omega;\mathbb{M}^p)$, the distribution $\mathbf{RA} \in D'(\Omega,\mathbb{M}^p)$ is well defined since $\mathbf{RA} \in L^\infty_{\rm loc}(\Omega,\mathbb{M}^p)$; likewise, each distribution $\mathbf{R}\partial_j \mathbf{A} \in D'(\Omega,\mathbb{M}^p)$ is well defined by the relations

$${}_{\mathcal{D}'(\Omega)}< \mathbf{R}\partial_j \mathbf{A}, \boldsymbol{\varphi} >_{\mathcal{D}(\Omega)} := {}_{H^{-1}(U)}< \partial_j \mathbf{A}, \mathbf{R}^T \boldsymbol{\varphi} >_{H^1_0(U)},$$

for all $\boldsymbol{\varphi} \in D(\Omega;\mathbb{M}^p)$, where U designates the interior of the support of $\boldsymbol{\varphi}$. The relations $\partial_j \mathbf{R} = \mathbf{RA}_j$ in $L^\infty_{\rm loc}(\Omega;\mathbb{M}^p)$ satisfied by the matrix fields $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega,\mathbb{O}^p)$ and $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega,\mathbb{A}^p)$ determined above therefore imply that

$$\partial_{ji}\mathbf{R} = (\partial_i \mathbf{R})\mathbf{A}_j + \mathbf{R}\partial_i \mathbf{A}_j = \mathbf{RA}_i\mathbf{A}_j + \mathbf{R}\partial_i \mathbf{A}_j \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^p),$$

$$\partial_{ij}\mathbf{R} = (\partial_j \mathbf{R})\mathbf{A}_i + \mathbf{R}\partial_j \mathbf{A}_i = \mathbf{RA}_j\mathbf{A}_i + \mathbf{R}\partial_j \mathbf{A}_i \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^p).$$

Hence the relations $\partial_{ji}\mathbf{R} = \partial_{ij}\mathbf{R}$ imply that

$$\mathbf{R}\partial_i \mathbf{A}_j - \mathbf{R}\partial_j \mathbf{A}_i + \mathbf{RA}_i\mathbf{A}_j - \mathbf{RA}_j\mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^p).$$

Consequently, for any matrix field $\boldsymbol{\varphi} \in D(\Omega;\mathbb{M}^p)$ with U as the interior of its support,

$${}_{H^{-1}(U)}< \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i, \mathbf{R}^T\boldsymbol{\varphi} >_{H^1_0(U)} = 0.$$

The matrices $\mathbf{R}^T(x)$ being invertible at each $x \in \Omega$ (since they are orthogonal), any matrix field $\boldsymbol{\psi} \in D(\Omega;\mathbb{M}^p)$ with U as the interior of its support can be written as $\boldsymbol{\psi} = \mathbf{R}^T\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathcal{D}(\Omega;\mathbb{M}^p)$, and $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the same support. Consequently,

$$\begin{aligned}&{}_{H^{-1}(U)} <\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i, \boldsymbol{\psi} >_{H^1_0(U)}\\ &\qquad = {}_{\mathcal{D}'(\Omega)}< \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j + \mathbf{A}_j\mathbf{A}_i, \boldsymbol{\psi} >_{\mathcal{D}(\Omega)} = 0.\end{aligned}$$

Since this relation thus holds for any matrix field $\boldsymbol{\psi} \in \mathcal{D}(\Omega;\mathbb{M}^p)$, the field $\mathbf{A}_j$ satisfy

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^p).$$

Because the components $\Gamma^k_{jl} = \partial_j \mathbf{g}_l \cdot \mathbf{g}^k$ of the matrix fields Γ_j may be also written as

$$\Gamma^k_{jl} = \frac{1}{2} g^{kr}(\partial_j g_{lr} + \partial_l g_{jr} - \partial_r g_{jl}),$$

the matrix fields $\boldsymbol{\Gamma}_j$ are also given in matrix form as

$$\boldsymbol{\Gamma}_j = \frac{1}{2}\mathbf{C}^{-1}(\partial_j \mathbf{C} + \boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T),$$

where $\mathbf{c}_j$ denotes the j-th column vector field of the field $\mathbf{C}$. Using this expression of the fields $\boldsymbol{\Gamma}_j$ in the expression of the fields $\mathbf{A}_j$ and noting that $\mathbf{U}\mathbf{C}^{-1} = \mathbf{U}^{-1}$ and $\partial_j \mathbf{C} = (\partial_j \mathbf{U})\mathbf{U} + \mathbf{U}\partial_j \mathbf{U}$, we finally obtain

$$\mathbf{A}_j = \frac{1}{2}\{\mathbf{U}^{-1}(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\}.$$

This expression shows that $\mathbf{A}_j(x)$ is an antisymmetric matrix at each $x \in \Omega$ and consequently, that $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^p)$. $\square$

4. A fundamental existence theorem for linear differential systems

Our proof in the next section that the compatibility relations shown in Section 3 to be necessarily satisfied by the matrix field $\mathbf{U} = (\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta})^{1/2}$ associated with a given immersion $\boldsymbol{\Theta}$ are also sufficient for the existence of the immersion $\boldsymbol{\Theta}$, relies in an essential way on the following *fundamental existence theorem for linear differential systems with little regularity*, which is due to S. Mardare [14, Theorem 3.6] (for smooth data, this theorem is a special case of earlier existence results of Cartan [1] and Thomas [19]). Recall that $p \geq 2$ is a given integer and that Latin indices range in $\{1, 2, \cdots, p\}$.

Theorem 4.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^p$ and let $q \geq 1$ be an integer. Let there be given matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^q)$, $\mathbf{B}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^p)$, and $\mathbf{C}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^{p\times q})$ that satisfy:*

$$\begin{aligned}
\partial_i\mathbf{A}_j + \mathbf{A}_i\mathbf{A}_j &= \partial_j\mathbf{A}_i + \mathbf{A}_j\mathbf{A}_i \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^q),\\
\partial_i\mathbf{B}_j + \mathbf{B}_j\mathbf{B}_i &= \partial_j\mathbf{B}_i + \mathbf{B}_i\mathbf{B}_j \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^p),\\
\partial_i\mathbf{C}_j + \mathbf{C}_i\mathbf{A}_j + \mathbf{B}_j\mathbf{C}_i &= \partial_j\mathbf{C}_i + \mathbf{C}_j\mathbf{A}_i + \mathbf{B}_i\mathbf{C}_j \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{p\times q}),
\end{aligned}$$

and let a point $x_0 \in \Omega$ and a matrix $\mathbf{F}_0 \in \mathbb{M}^{p\times q}$ be given. Then there exists one and only matrix field $\mathbf{Y} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{p\times q})$ that satisfies:

$$\begin{aligned}
\partial_j\mathbf{Y} &= \mathbf{Y}\mathbf{A}_j + \mathbf{B}_j\mathbf{Y} + \mathbf{C}_j \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{p\times q}),\\
\mathbf{Y}(x_0) &= \mathbf{F}_0.
\end{aligned}$$

As shown by S. Mardare, this existence result can be extended to the space $W^{1,\infty}(\Omega; \mathbb{M}^{p\times q})$ (cf. Theorem 4.2), but in order to state this extension, we first need some definitions.

In what follows, Ω designates any connected subset of $\mathbb{R}^p$. Given two points $x, y \in \Omega$, a *path joining x to y in Ω* is any mapping $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^p)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$ (there always exist such paths), and the *length* of such a path is defined by

$$L(\boldsymbol{\gamma}) := \int_0^1 \left|\frac{d\gamma}{dt}(t)\right| dt.$$

The *geodesic distance between two points* $x, y \in \Omega$ is then defined by

$$d_\Omega(x, y) = \inf\{L(\boldsymbol{\gamma}); \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y \text{ in } \Omega\},$$

and finally, the *geodesic diameter* of Ω is defined by

$$D_\Omega = \sup_{x,y\in\Omega} d_\Omega(x, y).$$

Note that $D_\Omega = +\infty$ is not excluded and that Ω is bounded if $D_\Omega < +\infty$. Otherwise it is easily seen (cf., e.g., [6, Lemma 2.3]) that any bounded connected subset of $\mathbb{R}^p$ with a Lipschitz-continuous boundary has a finite geodesic diameter.

As a complement to Theorem 4.1, we then have (the proof of this extension is analogous to that of [14, Corollary 3.4]):

Theorem 4.2. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^p$ whose geodesic diameter is finite, and let $q \geq 1$ be an integer. Let there be given matrix fields $\mathbf{A}_i \in L^\infty(\Omega;\mathbb{M}^q), \mathbf{B}_i \in L^\infty(\Omega;\mathbb{M}^p)$, and $\mathbf{C}_i \in L^\infty(\Omega;\mathbb{M}^{p\times q})$ that satisfy*

$$\begin{aligned}\partial_i\mathbf{A}_j + \mathbf{A}_i\mathbf{A}_j &= \partial_j\mathbf{A}_i + \mathbf{A}_j\mathbf{A}_i \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^q),\\ \partial_i\mathbf{B}_j + \mathbf{B}_j\mathbf{B}_i &= \partial_j\mathbf{B}_i + \mathbf{B}_i\mathbf{B}_j \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^p),\\ \partial_i\mathbf{C}_j + \mathbf{C}_i\mathbf{A}_j + \mathbf{B}_j\mathbf{C}_i &= \partial_j\mathbf{C}_i + \mathbf{C}_j\mathbf{A}_i + \mathbf{B}_i\mathbf{C}_j \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{p\times q}),\end{aligned}$$

and let a point $x_0 \in \Omega$ and a matrix $\mathbf{F}_0 \in \mathbb{M}^{p\times q}$ be given. Then there exists one and only matrix field $\mathbf{Y} \in W^{1,\infty}(\Omega;\mathbb{M}^{p\times q})$ that satisfies:

$$\begin{aligned}\partial_j\mathbf{Y} &= \mathbf{Y}\mathbf{A}_j + \mathbf{B}_j\mathbf{Y} + \mathbf{C}_j \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{p\times q}),\\ \mathbf{Y}(x_0) &= \mathbf{F}_0.\end{aligned}$$

It is worth noticing that both Theorems 4.1 and 4.2 contain two important special cases, viz., a *generalized Poincaré lemma*, corresponding to $\mathbf{A}_i = \mathbf{0}$ and $\mathbf{B}_i = \mathbf{0}$, and a *general existence theorem for Pfaff systems*, corresponding to $\mathbf{B}_i = \mathbf{0}$ and $\mathbf{C}_i = \mathbf{0}$.

5. Sufficiency of the compatibility relations

Under the assumption that the open set Ω is simply-connected, we now show that, if a symmetric and positive-definite matrix field $\mathbf{U}$ defined on Ω satisfies the compatibility relations that were found to be necessary in Theorem 3.1, then conversely there exists an immersion $\mathbf{\Theta} : \Omega \to \mathbb{R}^p$ such that $\mathbf{U} = (\boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta})^{1/2}$. Note that this existence result holds for fields $\mathbf{U}$ with little regularity.

Theorem 5.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^p$ and let there be given a matrix field $\mathbf{U} \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>)$ that satisfies*

$$\partial_i\mathbf{A}_j - \partial_j\mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega;\mathbb{A}^p),$$

where the matrix fields $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega;\mathbb{A}^p)$ are defined in terms of the matrix field $\mathbf{U}$ by

$$\mathbf{A}_j := \frac{1}{2}\{\mathbf{U}^{-1}(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\},$$

the field $\mathbf{c}_j \in W^{1,\infty}_{\rm loc}(\Omega,\mathbb{R}^p)$ denoting the j-th column vector field of the matrix field $\mathbf{U}^2 \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>)$.

Let there be given a point $x_0 \in \Omega$, a vector $\mathbf{a}_0 \in \mathbb{R}^p$, and a matrix $\mathbf{F}_0 \in \mathbb{M}^p$ that satisfies $(\mathbf{F}_0^T\mathbf{F}_0)^{1/2} = \mathbf{U}(x_0)$.

Then there exists one and only one immersion $\mathbf{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^p)$ that satisfies

$$\begin{aligned}\mathbf{U} &= (\boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta})^{1/2} \text{ in } W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^p_>),\\ \mathbf{\Theta}(x_0) &= \mathbf{a}_0 \text{ and } \boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{F}_0.\end{aligned}$$

Proof. We proceed in three stages. Note that the existence result of part (i) below uses only the compatibility relations satisfied by the matrix fields $\mathbf{A}_j$, i.e., it holds irrespective of the specific expression of the matrix fields $\mathbf{A}_j$ in terms of the matrix field $\mathbf{U}$.

(i) *Let there be given a point $x_0 \in \Omega$ and a matrix $\mathbf{R}_0 \in \mathbb{O}^p$. Then there exists one and only one matrix field $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{O}^p)$ that satisfies*

$$\begin{aligned} \partial_j \mathbf{R} &= \mathbf{R}\mathbf{A}_j \text{ in } L^\infty_{\rm loc}(\Omega; \mathbb{M}^p), \\ \mathbf{R}(x_0) &= \mathbf{R}_0. \end{aligned}$$

Since the matrix fields $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega; \mathbb{A}^p)$ satisfy

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_j \mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{A}^p),$$

the above Pfaff system has one and only one solution $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{M}^p)$ by Theorem 4.1.

The matrix field $\mathbf{R}^T\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega, \mathbb{M}^p)$ satisfies the differential system

$$\begin{aligned} \partial_j (\mathbf{R}^T\mathbf{R}) &= (\partial_j \mathbf{R})^T \mathbf{R} + \mathbf{R}^T \partial_j \mathbf{R} = \mathbf{A}_j^T (\mathbf{R}^T\mathbf{R}) + (\mathbf{R}^T\mathbf{R})\mathbf{A}_j \text{ in } L^\infty_{\rm loc}(\Omega; \mathbb{M}^p), \\ (\mathbf{R}^T\mathbf{R})(x_0) &= \mathbf{I}, \end{aligned}$$

which has one and only one solution, again by Theorem 4.1. Observing that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ is a solution of this system, we conclude that $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{O}^p)$.

(ii) As a preparation to part (iii), we note that *the matrix fields*

$$\mathbf{A}_j := \frac{1}{2}\{\mathbf{U}^{-1}(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j \mathbf{U} - (\partial_j \mathbf{U})\mathbf{U}^{-1}\}$$

may be also written as

$$\mathbf{A}_j = \mathbf{U}\boldsymbol{\Gamma}_j \mathbf{U}^{-1} - (\partial_j \mathbf{U})\mathbf{U}^{-1},$$

where the matrix fields $\boldsymbol{\Gamma}_j \in L^\infty_{\rm loc}(\Omega, \mathbb{M}^p)$ are defined by

$$\boldsymbol{\Gamma}_j := \frac{1}{2}\mathbf{U}^{-2}(\partial_j(\mathbf{U}^2) + \boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T).$$

This re-rewriting relies on a direct computation, which is omitted as it is straightforward.

(iii) *The matrix field $\mathbf{R} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{O}^p)$ being that determined in* (i)*, there exists one and only one vector field $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega, \mathbb{R}^p)$ that satisfies*

$$\begin{aligned} \boldsymbol{\nabla}\boldsymbol{\Theta} &= \mathbf{R}\mathbf{U} \text{ in } W^{1,\infty}_{\rm loc}(\Omega; \mathbb{M}^p), \\ \boldsymbol{\Theta}(x_0) &= \mathbf{a}_0. \end{aligned}$$

To begin with, we note that solving $\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{R}\mathbf{U}$ is the same as solving

$$\begin{aligned} \partial_j \boldsymbol{\Theta} &= \mathbf{R}\mathbf{u}_j \text{ in } W^{1,\infty}_{\rm loc}(\Omega; \mathbb{R}^p), \\ \boldsymbol{\Theta}(x_0) &= \boldsymbol{\Theta}_0, \end{aligned}$$

where $\mathbf{u}_j \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{R}^p)$ denotes the j-th column vector field of the matrix field $\mathbf{U}$. Resorting again to Theorem 4.1, we conclude that this system has one and only one solution $\boldsymbol{\Theta} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{R}^p)$ if the compatiblity relations

$$\partial_i(\mathbf{R}\mathbf{u}_j) = \partial_j(\mathbf{R}\mathbf{u}_i)$$

are satisfied (that $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega; \mathbb{R}^p)$ in turn clearly follows, since both fields $\mathbf{R}$ and $\mathbf{U}$ are in the space $W^{1,\infty}_{\rm loc}(\Omega, \mathbb{M}^p)$). Since $\partial_i \mathbf{R} = \mathbf{R}\mathbf{A}_i$ (cf. part (i)), these conditons take the equivalent form

$$\mathbf{A}_i \mathbf{u}_j + \partial_i \mathbf{u}_j = \mathbf{A}_j \mathbf{u}_i + \partial_j \mathbf{u}_i,$$

which, thanks to the specific expression of the matrix fields $\mathbf{A}_j$ in terms of the matrix fields $\boldsymbol{\Gamma}_j$ (see part (ii)), are seen after some straightforward computations to hold if, for all (i, j), the j-th column vector field of the matrix field $\mathbf{\Gamma}_i$ coincides with the i-th column vector field of the matrix field $\mathbf{\Gamma}_j$. This equality of vector fields itself immediately follows from the (already noted) observation that the elements Γ^k_{jl} (recall that k and l are respectively the row and column indices) of the matrix fields $\mathbf{\Gamma}_j$ can be also written as

$$\Gamma^k_{jl} = \frac{1}{2} g^{kr}(\partial_j g_{lr} + \partial_l g_{jr} - \partial_r g_{jl}),$$

where $(g^{kl}) := \mathbf{U}^{-2}$ and $(g_{ij}) := \mathbf{U}^2$.

It remains to satisfy the "initial" condition $\boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{F}_0$. This is achieved by letting $\mathbf{R}_0 := \mathbf{F}_0(\mathbf{F}_0^T\mathbf{F}_0)^{-1/2}$ in part (i). □

Naturally, if no "initial" conditions such as $\mathbf{\Theta}(x_0) = \mathbf{a}_0$ and $\boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{F}_0$ are imposed, the immersions $\mathbf{\Theta} \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^p)$ are then *uniquely defined up to isometries of* $\mathbb{R}^p$ *only*, according to the familiar *rigidity theorem* (which holds for immersions in the space $\mathcal{C}^1(\Omega; \mathbb{R}^p)$; cf., e.g., [3, Theorem 1.7-1]).

An inspection of the above proof immediately leads to the following existence result in the spaces $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^p), m \geq 1$.

Theorem 5.2. *Assume in* Theorem 5.1 *that the matrix field* $\mathbf{U}$ *belongs to the set* $\mathcal{C}^m(\Omega; \mathbb{S}^p_>)$ *for some integer* $m \geq 1$*, all the other assumptions and definitions of* Theorem 5.1 *holding* verbatim. *Then the immersion* $\mathbf{\Theta}$ *found in* Theorem 5.1 *belongs to the space* $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^p)$.

Under an additional assumption on the set Ω, a similar existence result holds in the space $W^{2,\infty}(\Omega, \mathbb{R}^p)$. We recall that the geodesic diameter of an open subset of $\mathbb{R}^p$ is defined in Section 4.

Theorem 5.3. *Assume in* Theorem 5.1 *that the geodesic diameter of* Ω *is finite and that the matrix field* $\mathbf{U}$ *belongs to the set* $W^{1,\infty}(\Omega; \mathbb{S}^p_>)$*, all the other assumptions and definitions of* Theorem 5.1 *holding verbatim. Then the immersion* $\mathbf{\Theta}$ *found in* Theorem 5.1 *belongs to the space* $W^{2,\infty}(\Omega, \mathbb{R}^p)$.

Proof. The proof is analogous to that of Theorem 5.1, save that the existence result of Theorem 4.1 is now replaced by that of Theorem 4.2. □

6. Special case of an open subset of $\mathbb{R}^3$

In this section, the dimension p of the underlying space is equal to three, which means that Latin indices range in $\{1, 2, 3\}$. In this case, the sufficient compatibility relations of Theorem 5.1 can be re-written in a remarkably simple and concise form, in terms of the matrix operators **CURL** and **COF** (whose definitions are recalled in Section 2) applied to an *ad hoc* matrix field $\mathbf{\Lambda}$, itself a function of the given matrix field $\mathbf{U}$. These relations are due to C. Vallée [20], who showed that they are *necessarily* satisfied by the matrix field $\mathbf{U} = (\boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta})^{1/2}$ associated with a *given* immersion $\mathbf{\Theta}$. We now show that they are also *sufficient*, according to the following global existence result in the space $W^{2,\infty}_{\text{loc}}(\Omega, \mathbb{R}^3)$ (similar existence results hold in the spaces $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^3), m \geq 1$, and $W^{2,\infty}(\Omega, \mathbb{R}^3)$; cf. Theorems 6.2 and 6.3).

Theorem 6.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^3$ and let there be given a matrix field $\mathbf{U} \in W^{1,\infty}_{\rm loc}(\Omega, \mathbb{S}^3_>)$ that satisfies*

$$\mathbf{CURL}\ \mathbf{\Lambda} + \mathbf{COF}\ \mathbf{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3),$$

where the matrix field $\mathbf{\Lambda} \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^3)$ is defined in terms of the field $\mathbf{U}$ by

$$\mathbf{\Lambda} = \frac{1}{\det \mathbf{U}} \Big\{ \mathbf{U}\ (\mathbf{CURL}\ \mathbf{U})^T \mathbf{U} - \frac{1}{2} (\mathrm{tr}[\mathbf{U}\ (\mathbf{CURL}\ \mathbf{U})^T]) \mathbf{U} \Big\}.$$

Let there be given a point $x_0 \in \Omega$, a vector $\mathbf{a}_0 \in \mathbb{R}^3$, and a matrix $\mathbf{F}_0 \in \mathbb{M}^3$ that satisfies $(\mathbf{F}_0^T \mathbf{F}_0)^{1/2} = \mathbf{U}(x_0)$.

Then there exists one and only one immersion $\mathbf{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega; \mathbb{R}^3)$ that satisfies

$$\mathbf{U} = (\boldsymbol{\nabla}\mathbf{\Theta}^T \boldsymbol{\nabla}\mathbf{\Theta})^{1/2} \text{ in } W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>),$$
$$\mathbf{\Theta}(x_0) = \mathbf{a}_0 \text{ and } \boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{F}_0.$$

Proof. For the sake of clarity, this proof is broken into five parts, numbered (i) to (v). In what follows, the notation $[\mathbf{A}]_j : \Omega \to \mathbb{R}^3$ designates the j-th column vector field of a given matrix field $\mathbf{A}$.

(i) Given matrix fields $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega; \mathbb{A}^3)$, thus of the form

$$\mathbf{A}_j = \begin{pmatrix} 0 & -a_{3j} & a_{2j} \\ a_{3j} & 0 & -a_{1j} \\ -a_{2j} & a_{1j} & 0 \end{pmatrix},$$

define the matrix field $\mathbf{\Lambda} \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^3)$ by

$$\mathbf{\Lambda} := (a_{ij}).$$

Equivalently,

$$[\mathbf{\Lambda}]_j \wedge \mathbf{v} = \mathbf{A}_j \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{R}^3.$$

Our first objective consists in showing that *the relations assumed in* Theorem 5.1 *on the matrix fields $\mathbf{A}_j$ are equivalent to the relation assumed in* Theorem 6.1 *on the matrix field* $\mathbf{\Lambda}$. To this end, a direct computation shows that, given any pair $(i,j) = (k, k+1)$ where $k \in \{1,2,3\}$ and $(k+1)$ is counted *modulo* 3, the relation

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_j \mathbf{A}_i = \mathbf{0}$$

found in Theorem 5.1 is satisfied if and only if

$$(\mathbf{CURL}\ \mathbf{\Lambda})_{l,k+2} + (\mathbf{COF}\ \mathbf{\Lambda})_{l,k+2} = 0 \qquad \text{for all } l \in \{1,2,3\},$$

where $(k+2)$ is counted *modulo* 3. Hence *the relations*

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_j \mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3),$$

which hold for all $i, j \in \{1,2,3\}$ if and only if they hold for $(i,j) = (k, k+1), k \in \{1,2,3\}$, *are satisfied if and only if*

$$\mathbf{CURL}\ \mathbf{\Lambda} + \mathbf{COF}\ \mathbf{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3).$$

(ii) Given a matrix field $\mathbf{C} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$, define the matrix fields $\mathbf{A}_j$ as in Theorem 5.1, i.e., by

$$\mathbf{A}_j := (\mathbf{U}\mathbf{\Gamma}_j - \partial_j \mathbf{U})\mathbf{U}^{-1} \in L^\infty_{\rm loc}(\Omega; \mathbb{A}^3),$$

where

$$\mathbf{U} := \mathbf{C}^{1/2} \text{ and } \mathbf{\Gamma}_j := \frac{1}{2}\mathbf{C}^{-1}(\partial_j \mathbf{C} + \boldsymbol{\nabla} \mathbf{c}_j - (\boldsymbol{\nabla} \mathbf{c}_j)^T) \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^3) \text{ and } \mathbf{c}_j := [\mathbf{C}]_j,$$

and let $\mathbf{\Lambda} \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^3)$ denote the matrix field defined by the relations

$$\mathbf{a}_j \wedge \mathbf{v} = \mathbf{A}_j \mathbf{v} \text{ for all } \mathbf{v} \in \mathbb{R}^3 \text{ where } \mathbf{a}_j := [\mathbf{\Lambda}]_j$$

(this definition makes sense since the matrix fields $\mathbf{A}_j$ are antisymmetric).

Our second objective consists in showing that *the matrix field* $\mathbf{\Lambda}$ *is then given in terms of the matrix field* $\mathbf{U}$ *by the expression announced in the statement of* Theorem 6.1.

Before doing so (in part (iii)), we begin by listing a series of formulas that will be needed for this purpose (these formulas are of course only valid for vector and matrix fields with sufficient smoothness, not specified here for conciseness, but which should otherwise be clear in each instance): *Given a matrix field* $\mathbf{A}$,

$$\det \mathbf{A} = (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot \mathbf{a}_3 = \mathbf{a}_1 \cdot (\mathbf{a}_2 \wedge \mathbf{a}_3) \text{ where } \mathbf{a}_i := [\mathbf{A}]_i.$$

Given vector fields $\mathbf{b}_i$,

$$(\mathbf{b}_1 \wedge \mathbf{b}_2) \wedge \mathbf{b}_3 = (\mathbf{b}_1 \cdot \mathbf{b}_3)\mathbf{b}_2 - (\mathbf{b}_3 \cdot \mathbf{b}_2)\mathbf{b}_1.$$

Given an invertible matrix field $\mathbf{U}$*, let* $\mathbf{u}_j := [\mathbf{U}]_j$ *and* $\mathbf{v}_j := [\mathbf{U}^{-1}]_j$*; then*

$$\begin{aligned}
&\mathbf{u}_1 = (\det \mathbf{U})\mathbf{v}_2 \wedge \mathbf{v}_3, \qquad \mathbf{u}_2 = (\det \mathbf{U})\mathbf{v}_3 \wedge \mathbf{v}_1, \qquad \mathbf{u}_3 = (\det \mathbf{U})\mathbf{v}_1 \wedge \mathbf{v}_2,\\
&\mathbf{v}_1 = \frac{1}{\det \mathbf{U}}\mathbf{u}_2 \wedge \mathbf{u}_3, \qquad \mathbf{v}_2 = \frac{1}{\det \mathbf{U}}\mathbf{u}_3 \wedge \mathbf{u}_1, \qquad \mathbf{v}_3 = \frac{1}{\det \mathbf{U}}\mathbf{u}_1 \wedge \mathbf{u}_2
\end{aligned}$$

(the first formula above is well known; the others are immediately verified)

Next, *given a matrix field* $\mathbf{U}$*, let* $\mathbf{u}_j := [\mathbf{U}]_j$*; then*

$$\begin{aligned}
&\operatorname{tr}[\mathbf{U}(\mathbf{CURL\ U})^T] = (\partial_2 \mathbf{u}_3 - \partial_3 \mathbf{u}_2) \cdot \mathbf{u}_1 + (\partial_3 \mathbf{u}_1 - \partial_1 \mathbf{u}_3) \cdot \mathbf{u}_2 + (\partial_1 \mathbf{u}_2 - \partial_2 \mathbf{u}_1) \cdot \mathbf{u}_3,\\
&\mathbf{U}(\mathbf{CURL\ U})^T \mathbf{u}_j = [(\partial_2 \mathbf{u}_3 - \partial_3 \mathbf{u}_2) \cdot \mathbf{u}_j]\mathbf{u}_1 + [(\partial_3 \mathbf{u}_1 - \partial_1 \mathbf{u}_3) \cdot \mathbf{u}_j]\mathbf{u}_2 + [(\partial_1 \mathbf{u}_2 - \partial_2 \mathbf{u}_1) \cdot \mathbf{u}_j]\mathbf{u}_3
\end{aligned}$$

(these two formulas are straightforward consequences of the relations $[\mathbf{CURL\ U}]_1 = \partial_2 \mathbf{u}_3 - \partial_3 \mathbf{u}_2, [\mathbf{CURL\ U}]_2 = \partial_3 \mathbf{u}_1 - \partial_1 \mathbf{u}_3,$ and $[\mathbf{CURL\ U}]_3 = \partial_1 \mathbf{u}_2 - \partial_2 \mathbf{u}_1$).

Finally, *given an invertible matrix field* $\mathbf{U}$*, let again* $\mathbf{u}_j := [\mathbf{U}]_j$ *and* $\mathbf{v}_j := [\mathbf{U}^{-1}]_j$*; then, for all* $i \in \{1, 2, 3\}$,

$$(\partial_i \mathbf{u}_j - \partial_j \mathbf{u}_i) \wedge \mathbf{v}_j = \frac{1}{\det \mathbf{U}}\Big\{\mathbf{U}(\mathbf{CURL\ U})^T \mathbf{u}_i - (\operatorname{tr}[\mathbf{U}(\mathbf{CURL\ U})^T])\mathbf{u}_i\Big\}.$$

To prove this last relation, we note that, thanks to the previous formulas,

$$\begin{aligned}
(\det \mathbf{U})(\partial_i \mathbf{u}_j - \partial_j \mathbf{u}_i) \wedge \mathbf{v}_j &= (\partial_i \mathbf{u}_1 - \partial_1 \mathbf{u}_i) \wedge (\mathbf{u}_2 \wedge \mathbf{u}_3)\\
&\quad + (\partial_i \mathbf{u}_2 - \partial_2 \mathbf{u}_i) \wedge (\mathbf{u}_3 \wedge \mathbf{u}_1) + (\partial_i \mathbf{u}_3 - \partial_3 \mathbf{u}_i) \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2)\\
&= (\mathbf{u}_3 \wedge \mathbf{u}_2) \wedge (\partial_i \mathbf{u}_1 - \partial_1 \mathbf{u}_i)\\
&\quad + (\mathbf{u}_1 \wedge \mathbf{u}_3) \wedge (\partial_i \mathbf{u}_2 - \partial_2 \mathbf{u}_i) + (\mathbf{u}_2 \wedge \mathbf{u}_1) \wedge (\partial_i \mathbf{u}_3 - \partial_3 \mathbf{u}_i)\\
&= ([\partial_i \mathbf{u}_1 - \partial_1 \mathbf{u}_i] \cdot \mathbf{u}_3)\mathbf{u}_2 - ([\partial_i \mathbf{u}_1 - \partial_1 \mathbf{u}_i] \cdot \mathbf{u}_2)\mathbf{u}_3\\
&\quad + ([\partial_i \mathbf{u}_2 - \partial_2 \mathbf{u}_i] \cdot \mathbf{u}_1)\mathbf{u}_3 - ([\partial_i \mathbf{u}_2 - \partial_2 \mathbf{u}_i] \cdot \mathbf{u}_3)\mathbf{u}_1\\
&\quad + ([\partial_i \mathbf{u}_3 - \partial_3 \mathbf{u}_i] \cdot \mathbf{u}_2)\mathbf{u}_1 - ([\partial_i \mathbf{u}_3 - \partial_3 \mathbf{u}_i] \cdot \mathbf{u}_1)\mathbf{u}_2\\
&= \mathbf{U}(\mathbf{CURL\ U})^T \mathbf{u}_i - (\operatorname{tr}[\mathbf{U}(\mathbf{CURL\ U})^T])\mathbf{u}_i.
\end{aligned}$$

(iii) Let each matrix field $\mathbf{A}_i$ be defined as in (ii) in terms of the matrix field $\mathbf{U}$, so that $\mathbf{A}_i\mathbf{U} = \mathbf{U}\boldsymbol{\Gamma}_i - \partial_i\mathbf{U}$, or equivalently, $\mathbf{A}_i\mathbf{u}_j = \mathbf{U}[\boldsymbol{\Gamma}_i]_j - \partial_i\mathbf{u}_j$ for all j, where $\mathbf{u}_j := [\mathbf{U}]_j$. Consequently, for all j,

$$\mathbf{A}_i\mathbf{u}_j - \mathbf{A}_j\mathbf{u}_i = \partial_j\mathbf{u}_i - \partial_i\mathbf{u}_j,$$

since $[\boldsymbol{\Gamma}_i]_j = [\boldsymbol{\Gamma}_j]_i$. Hence, again for all j,

$$\mathbf{a}_i \wedge \mathbf{u}_j - \mathbf{a}_j \wedge \mathbf{u}_i = \partial_j\mathbf{u}_i - \partial_i\mathbf{u}_j,$$

by definition of the vector fields $\mathbf{a}_i$. We thus have, by part (ii),

$$\begin{aligned}(\mathbf{a}_i \wedge \mathbf{u}_j) \wedge \mathbf{v}_j &- (\mathbf{a}_j \wedge \mathbf{u}_i) \wedge \mathbf{v}_j\\ &= (\mathbf{a}_i \cdot \mathbf{v}_j)\mathbf{u}_j - (\mathbf{u}_j \cdot \mathbf{v}_j)\mathbf{a}_i - (\mathbf{a}_j \cdot \mathbf{v}_j)\mathbf{u}_i + (\mathbf{u}_i \cdot \mathbf{v}_j)\mathbf{a}_j\\ &= (\mathbf{a}_i \cdot \mathbf{v}_j)\mathbf{u}_j - \mathbf{a}_i - (\mathbf{a}_j \cdot \mathbf{v}_j)\mathbf{u}_i\\ &= (\partial_j\mathbf{u}_i - \partial_i\mathbf{u}_j) \wedge \mathbf{v}_j \text{ (no summation on j)},\end{aligned}$$

which in turn implies that, for all $j \neq i$,

$$\mathbf{a}_i = (\mathbf{a}_i \cdot \mathbf{v}_j)\mathbf{u}_j - (\mathbf{a}_j \cdot \mathbf{v}_j)\mathbf{u}_i + (\partial_i\mathbf{u}_j - \partial_j\mathbf{u}_i) \wedge \mathbf{v}_j \text{ (no summation on j)}.$$

This gives

$$2\mathbf{a}_i = \sum_{j\neq i}(\mathbf{a}_i \cdot \mathbf{v}_j)\mathbf{u}_j - \sum_{j\neq i}(\mathbf{a}_j \cdot \mathbf{v}_j)\mathbf{u}_i + \sum_{j\neq i}(\partial_i\mathbf{u}_j - \partial_j\mathbf{u}_i) \wedge \mathbf{v}_j,$$

or equivalently (since $\mathbf{a}_i = (\mathbf{a}_i \cdot \mathbf{v}_j)\mathbf{u}_j$),

$$\mathbf{a}_i = -(\mathbf{a}_j \cdot \mathbf{v}_j)\mathbf{u}_i + (\partial_i\mathbf{u}_j - \partial_j\mathbf{u}_i) \wedge \mathbf{v}_j.$$

Therefore,

$$(\mathbf{a}_i \cdot \mathbf{v}_i) = -3(\mathbf{a}_j \cdot \mathbf{v}_j) + (\partial_j\mathbf{u}_i - \partial_i\mathbf{u}_j) \cdot (\mathbf{v}_i \wedge \mathbf{v}_j).$$

Using again part (ii), we thus obtain

$$\begin{aligned}2(\mathbf{a}_i \cdot \mathbf{v}_i) &= (\partial_1\mathbf{u}_2 - \partial_2\mathbf{u}_1) \cdot (\mathbf{v}_2 \wedge \mathbf{v}_1) + (\partial_2\mathbf{u}_3 - \partial_3\mathbf{u}_2) \cdot (\mathbf{v}_3 \wedge \mathbf{v}_2)\\ &\qquad + (\partial_3\mathbf{u}_1 - \partial_1\mathbf{u}_3) \cdot (\mathbf{v}_1 \wedge \mathbf{v}_3)\\ &= -\frac{1}{\det\mathbf{U}}[(\partial_1\mathbf{u}_2 - \partial_2\mathbf{u}_1) \cdot \mathbf{u}_3 + (\partial_2\mathbf{u}_3 - \partial_3\mathbf{u}_2) \cdot \mathbf{u}_1 + (\partial_3\mathbf{u}_1 - \partial_1\mathbf{u}_3) \cdot \mathbf{u}_2]\\ &= -\frac{1}{\det\mathbf{U}}\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T].\end{aligned}$$

To sum up, we have shown that

$$\mathbf{a}_i = (\partial_i\mathbf{u}_j - \partial_j\mathbf{u}_i) \wedge \mathbf{v}_j + \frac{1}{2\det\mathbf{U}}(\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T])\mathbf{u}_i.$$

This relation, combined with the last formula from part (ii), in turn implies that

$$\mathbf{a}_i = \frac{1}{\det\mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T\mathbf{u}_i - \frac{1}{2}(\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T])\mathbf{u}_i\right\},$$

or equivalently, in matrix form,

$$\boldsymbol{\Lambda} = \frac{1}{\det\mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T])\mathbf{U}\right\}.$$

We have thus found an explicit expression of the matrix field $\boldsymbol{\Lambda}$ in terms of the matrix field $\mathbf{U}$, as desired.

(iv) Conversely, given a matrix field $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{S}^3_>)$, define the matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$ by the last formula above, and let the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3)$ be defined by the relations

$$\mathbf{A}_j\mathbf{v} = \mathbf{a}_j \wedge \mathbf{v} \text{ for all } \mathbf{v} \in \mathbb{R}^3 \text{ where } \mathbf{a}_j := [\boldsymbol{\Lambda}]_j.$$

Our third objective consists in showing that *the matrix fields* $\mathbf{A}_j$ *are given by*

$$\mathbf{A}_j = (\mathbf{U}\boldsymbol{\Gamma}_j - \partial_j \mathbf{U})\mathbf{U}^{-1},$$

where

$$\boldsymbol{\Gamma}_j = \frac{1}{2}\mathbf{U}^{-2}(\partial_j(\mathbf{U}^2) + \boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T) \text{ and } \mathbf{c}_j := [\mathbf{U}^2]_j.$$

We claim that this conclusion can be reached without any further computation, by means of the following simple argument. Recall that, from our convention set up in Section 2, the point values of a function in $L^\infty_{\mathrm{loc}}(\Omega)$ are well-defined real numbers. So, given a matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$, let the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3)$ be defined as above. Now, at each point $x \in \Omega$, the linear mapping

$$(\mathbf{A}_1(x), \mathbf{A}_2(x), \mathbf{A}_3(x)) \in (\mathbb{A}^3)^3 \to \boldsymbol{\Lambda}(x) \in \mathbb{M}^3$$

defined by the relations $[\boldsymbol{\Lambda}(x)]_j \wedge \mathbf{v} = \mathbf{A}_j(x)\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$ is clearly one-to-one and onto between two finite-dimensional linear spaces of dimension nine. This observation shows that, given any field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$, there is one and only one field $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \in (L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3))^3$ that satisfies $\mathbf{A}_j\mathbf{v} = \mathbf{a}_j \wedge \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. The assertion thus follows from the computations made in part (iii).

(v) The existence and uniqueness of the immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ follow by combining the equivalences established in part (i) and in parts (iii)-(iv) with Theorem 5.1. $\square$

As expected, part (iv) of the above proof can be also established by means of a direct (although somewhat delicate) computation, which incidentally produces the following, interesting *per se*, identity, valid for any matrix field $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$ and all $i \in \{1, 2, 3\}$:

$$\begin{aligned}&\frac{1}{2}\mathbf{U}^{-2}(\partial_i(\mathbf{U}^2) + \boldsymbol{\nabla}\mathbf{c}_i - (\boldsymbol{\nabla}\mathbf{c}_i)^T)\\ &\qquad = \mathbf{U}^{-1}\left(\partial_i\mathbf{U} + \frac{1}{\det\mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T - \frac{1}{2}\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\ \mathbf{U})^T]\mathbf{I}\right\}\mathbf{U}^\#_i\right),\end{aligned}$$

where each matrix field $\mathbf{U}^\#_i \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$ is defined by $[\mathbf{U}^\#_i]_j := \mathbf{u}_i \wedge \mathbf{u}_j$.

In the same manner that the existence result of Theorem 5.1 was extended in Theorems 5.2 and 5.3, that of Theorem 6.1 can be extended to the spaces $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^3)$, $m \geq 1$, and $W^{2,\infty}(\Omega; \mathbb{R}^3)$, as follows.

Theorem 6.2. *Assume in* Theorem 6.1 *that the matrix field* $\mathbf{U}$ *belongs to the set* $\mathcal{C}^m(\Omega; \mathbb{S}^3_>)$ *for some integer* $m \geq 1$*, all the other assumptions and definitions of* Theorem 6.1 *holding* verbatim. *Then the immersion* $\boldsymbol{\Theta}$ *found in* Theorem 6.1 *belongs to the space* $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^3)$.

Theorem 6.3. *Assume in* Theorem 6.1 *that the geodesic diameter of* Ω *is finite and that the matrix field* $\mathbf{U}$ *belongs to the set* $W^{1,\infty}(\Omega; \mathbb{S}^3_>)$*, all the other assumptions and definitions of* Theorem 6.1 *holding* verbatim. *Then the immersion* $\boldsymbol{\Theta}$ *found in* Theorem 6.1 *belongs to the space* $W^{2,\infty}(\Omega; \mathbb{R}^3)$.

To conclude our analysis, we show that, as expected, the compatibility relations found in Theorem 6.1 are equivalent to the vanishing of the Riemann curvature tensor.

Theorem 6.4. *Let* Ω *be an open subset of* $\mathbb{R}^3$. *Then a matrix field* $\mathbf{U} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$ *satisfies*

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3),$$

where the matrix field $\boldsymbol{\Lambda} \in L^\infty_{\rm loc}(\Omega, \mathbb{M}^3)$ *is defined in terms of the field* $\mathbf{U}$ *by*

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}} \Big\{ \mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T])\mathbf{U} \Big\}.$$

if and only if the matrix field $\mathbf{C} = (g_{ij}) := \mathbf{U}^2 \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$ *satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \mathcal{D}'(\Omega),$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \in L^\infty_{\rm loc}(\Omega) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq} \in L^\infty_{\rm loc}(\Omega),$$

where $(g^{pq}) := (g_{ij})^{-1} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$.

Proof. Since the equivalence between the two sets of compatibility relations is a "local" property, we assume without loss of generality that Ω is simply-connected. This being the case, assume that a matrix field $\mathbf{U} \in W^{1,\infty}_{\rm loc}(\Omega, \mathbb{S}^3_>)$ satisfies $\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\boldsymbol{\Lambda} = \mathbf{0}$ in $\mathcal{D}'(\Omega; \mathbb{M}^3)$, with the matrix field $\boldsymbol{\Lambda}$ defined as above in terms of $\mathbf{U}$. Then, by Theorem 6.1, there exists an immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega; \mathbb{R}^3)$ that satisfies $(\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta})^{1/2} = \mathbf{U}$ in $W^{1,\infty}_{\rm loc}(\Omega, \mathbb{S}^3_>)$. That the distributions $R_{qijk} \in \mathcal{D}'(\Omega)$ associated with the matrix field $(g_{ij}) := \mathbf{U}^2 = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ vanish is the well-known necessary condition recalled in Section 1.

Assume conversely that a matrix field $\mathbf{C} = (g_{ij}) \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$ is such that the associated distributions $R_{qijk} \in \mathcal{D}'(\Omega)$ vanish. By the existence theorem with little regularity of S. Mardare [14], there exists an immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega; \mathbb{R}^3)$ that satisfies $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \mathbf{C}$ in $W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$. By Theorem 3.1, the matrix fields $\mathbf{A}_j \in L^\infty_{\rm loc}(\Omega; \mathbb{A}^3)$ defined by

$$\mathbf{A}_j := \frac{1}{2}\{\mathbf{U}^{-1}(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\}$$

in terms of the matrix field $\mathbf{U} := \mathbf{C}^{1/2} \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^3_>)$, therefore satisfy

$$\partial_i\mathbf{A}_j - \partial_j\mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{A}^3).$$

But, as shown in the proof of Theorem 6.1, these relations are respectively equivalent to

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}} \Big\{ \mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}\,(\mathbf{CURL}\,\mathbf{U})^T])\mathbf{U} \Big\}$$

and to

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3).$$

Hence the two sets of compatibility relations are equivalent. □

Note that, by contrast with the proof given above, a proof by direct computation (i.e., without resorting to existence theorems) otherwise turns out to be surprisingly lengthy and delicate (see [21]).

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